

# WEIGHT FILTRATION ON THE COHOMOLOGY OF ALGEBRAIC VARIETIES

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**ABSTRACT.** We show that the étale cohomology (with compact supports) of an algebraic variety  $X$  over an algebraically closed field has the canonical weight filtration  $W$ , and prove that the middle weight part of the cohomology with compact supports of  $X$  is a subspace of the intersection cohomology of a compactification  $X'$  of  $X$ , or equivalently, the middle weight part of the (so-called) Borel-Moore homology of  $X$  is a quotient of the intersection cohomology of  $X'$ . We are informed that this has been shown by A. Weber in the case  $X$  is proper (and  $k = \mathbb{C}$ ) using a theorem of G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber and L. Kaup on morphisms between intersection complexes. We show that the assertion immediately follows from Gabber's purity theorem for intersection complexes.

## Introduction

Let  $X$  be an irreducible algebraic variety defined over an algebraically closed field  $k$ . Let  $H_j^{\text{BM}}(X)$  denote (so-called) Borel-Moore homology, i.e. the dual of  $H_c^j(X)$ . Here cohomology means étale cohomology with  $l$ -adic coefficients where  $l$  is a prime number different from  $\text{char } k$  (and similarly for cohomology with compact supports). In the case  $k = \mathbb{C}$ , we may also assume that it is singular cohomology with  $\mathbb{Q}$ -coefficients as in [1] using the comparison theorem. If  $\text{char } k = 0$  or  $k$  is the algebraic closure of a finite field, it is known that these cohomology and homology have the canonical weight filtration  $W$  so that  $H_j^{\text{BM}}(X)$  has weights  $\geq -j$ , see [2], [5], [7]. However, the general case of characteristic  $p > 0$  does not seem to be treated in the references. In this paper, we show that this easily follows from [2], [6] using a model, see (1.2).

Let  $\overline{X}$  be a compactification of  $X$ , and  $\text{IH}^j(\overline{X})$  denote intersection cohomology, see [2], [10]. Note that  $\text{IH}^j(\overline{X})$  is pure of weight  $j$  by Gabber's purity theorem [2] together with the stability of pure complexes by the direct image under a proper morphism [7]. We have a canonical morphism

$$\text{IH}^{2n-j}(\overline{X})(n) \rightarrow H_j^{\text{BM}}(X),$$

which is the dual of  $H_c^j(X) \rightarrow \text{IH}^j(\overline{X})$ , where  $n = \dim X$ , and  $(n)$  is the Tate twist. It was shown by G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber and L. Kaup (see [1]) that any algebraic cycle classes can be lifted (noncanonically) to the intersection cohomology by the above morphism at least when  $k = \mathbb{C}$ . In this paper we prove

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*Date:* June 15, 2006, v.2.

**Theorem 1.** *The canonical morphism  $\mathrm{IH}^{2n-j}(\overline{X})(n) \rightarrow \mathrm{Gr}_W^{-j} H_j^{\mathrm{BM}}(X)$  is surjective, or equivalently,  $\mathrm{Gr}_W^j H_c^j(X) \rightarrow \mathrm{IH}^j(\overline{X})$  is injective.*

This is a refinement of the theorem in [1] explained above, since the cycle class of an algebraic  $d$ -cycle is defined in  $(\mathrm{Gr}_W^{-2d} H_{2d}^{\mathrm{BM}}(X))(-d)$ . We are informed that the assertion has been proved by A. Weber [22] in the case  $X$  is proper (and  $k = \mathbb{C}$ ) using a theorem in [1] on morphisms between intersection complexes. In the case  $\overline{X}$  is smooth and  $k = \mathbb{C}$ , Theorem 1 follows from the construction of mixed Hodge structure in [5]. In general we have the surjectivity of the canonical morphism

$$\mathrm{IH}^{2n-j}(\overline{X})(n) \rightarrow \mathrm{Gr}_{-j}^W(\mathrm{IH}^{2n-j}(X)(n)),$$

factoring the first morphism in Theorem 1, see Remark (i) in (2.2). The proof of Theorem 1 immediately follows from Gabber's purity theorem for intersection complexes ([2], [3]). We also give a shorter proof of the theorem in [1] on morphisms between intersection complexes (here it is not necessary to assume  $k = \mathbb{C}$ ):

**Theorem 2** (see [1]). *Let  $f : X \rightarrow Y$  be a morphism of equidimensional varieties over an algebraically closed field  $k$ . Then there is a noncanonical morphism*

$$f^* \mathrm{IC}_Y \mathbb{Q}_l \rightarrow \mathrm{IC}_X \mathbb{Q}_l,$$

*whose composition with the canonical morphism  $\mathbb{Q}_l \rightarrow f^* \mathrm{IC}_Y \mathbb{Q}_l$  coincides with the canonical morphism  $\mathbb{Q}_l \rightarrow \mathrm{IC}_X \mathbb{Q}_l$ .*

In [1], this was proved by reducing to the case where  $X$  is a closed subvariety of codimension 1 in  $Y$ , and using induction on stratum, see also [21]. We show that Theorem 2 immediately follows from Gabber's purity Theorem. In the case of a closed embedding of irreducible varieties with relative dimension 1, they also showed the existence of a canonical morphism, see [1]. We give another proof of it using the weight filtration on mixed perverse sheaves [2], see (2.4).

If  $X, Y$  are irreducible,  $f$  is proper surjective, and  $\mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l$  is a shifted perverse sheaf on  $Y$  (e.g. if  $\dim X = \dim Y = 2$  or  $f$  is finite and surjective), then the semisimplicity of pure perverse sheaves [2] implies the canonical morphisms

$$\mathrm{IC}_Y \mathbb{Q}_l \rightarrow \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l, \quad \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l \rightarrow \mathrm{IC}_Y \mathbb{Q}_l.$$

This can be used to construct cohomological correspondences, see [8].

This paper is a consequence of our discussions on the first author's work [14] (see also [4]) at the conferences at Kagoshima (December 2004) and at Baltimore (March 2005). We thank the organizers of the conferences. We also thank H. Esnault and L. Illusie for useful comments and questions about the canonical choice of the morphisms, and O. Gabber for pointing out an error in an early version of this paper.

In Section 1 we show the existence of the weight filtration on the cohomology of a variety over any algebraically closed field. In Section 2 we prove the main theorems using Gabber's purity theorem. In Section 3 we give an application of the decomposition theorem of Beilinson, Bernstein and Deligne.

## 1. Preliminaries

**1.1. Intersection cohomology.** Let  $X$  be an algebraic variety (i.e. a separated reduced scheme of finite type) over an algebraically closed field. Let  $l$  be a prime different from the characteristic of  $k$ . By [2], [7], we have the bounded derived category with constructible cohomology  $D_c^b(X, \mathbb{Q}_l)$ . Then the category of perverse sheaves  $\text{Perv}(X, \mathbb{Q}_l)$  is defined to be a full subcategory of  $D_c^b(X, \mathbb{Q}_l)$  satisfying certain conditions, and it is an abelian category, see [2].

If  $X$  is purely  $n$ -dimensional, there is uniquely a shifted perverse sheaf  $\text{IC}_X \mathbb{Q}_l \in D_c^b(X, \mathbb{Q}_l)$  such that  $(\text{IC}_X \mathbb{Q}_l)[n] \in \text{Perv}(X, \mathbb{Q}_l)$ , the restriction of  $\text{IC}_X \mathbb{Q}_l$  to the smooth part  $U$  of  $X$  is the constant sheaf  $\mathbb{Q}_l$ , and  $(\text{IC}_X \mathbb{Q}_l)[n]$  has no nontrivial sub nor quotient objects in  $\text{Perv}(X, \mathbb{Q}_l)$  which are supported on the complement of  $U$ , see [2]. This is called the intersection complex. To simplify the notation, we use the above normalization of intersection complex as in [1], which is different from [2]. We have unique morphisms

$$(1.1.1) \quad \mathbb{Q}_l \rightarrow \text{IC}_X \mathbb{Q}_l, \quad \text{IC}_X \mathbb{Q}_l(n)[2n] \rightarrow \mathbb{D}\mathbb{Q}_l \quad \text{on } X,$$

which are dual of each other, and whose restriction to the smooth part  $U$  of  $X$  is the identity on  $\mathbb{Q}_l|_U$  or  $\mathbb{D}\mathbb{Q}_l|_U$ . Here  $(n)$  denotes the Tate twist,  $\mathbb{Q}_l$  is a constant sheaf on  $X$ , and  $\mathbb{D}\mathbb{Q}_l$  is the dual of  $\mathbb{Q}_l$  which is isomorphic to the dualizing complex  $a^! \mathbb{Q}_l$  where  $a : X \rightarrow \text{Spec } k$ . The first morphism of (1.1.1) follows from the fact that  ${}^p\mathcal{H}^j \mathbb{Q}_l = 0$  for  $j > n$  and  $(\text{IC}_X \mathbb{Q}_l)[n]$  is a canonical quotient of  ${}^p\mathcal{H}^n \mathbb{Q}_l$ . For the last assertion, note that  ${}^p\mathcal{H}^n \mathbb{Q}_l$  has no nontrivial quotient  $M$  whose support has dimension  $< n$ , where  ${}^p\mathcal{H}^j$  denotes the perverse cohomology sheaf. (Indeed, this follows from

$$\text{Hom}({}^p\mathcal{H}^n \mathbb{Q}_l, M) = \text{Hom}(\mathbb{Q}_l[n], M) = \text{Hom}(i^* \mathbb{Q}_l[n], i^* M) = 0,$$

by the vanishing of negative extension groups for perverse sheaves [2], where  $i$  denotes the inclusion of the support of  $M$  into  $X$ .)

We define intersection cohomology (with compact supports) by

$$\text{IH}^j(X, \mathbb{Q}_l) = H^j(X, \text{IC}_X \mathbb{Q}_l), \quad \text{IH}_c^j(X, \mathbb{Q}_l) = H_c^j(X, \text{IC}_X \mathbb{Q}_l).$$

We will denote their dual by  $\text{IH}_j(X, \mathbb{Q}_l)$  and  $\text{IH}_j^{\text{BM}}(X, \mathbb{Q}_l)$  respectively. These are isomorphic to  $\text{IH}_c^{2n-j}(X, \mathbb{Q}_l)(n)$  and  $\text{IH}^{2n-j}(X, \mathbb{Q}_l)(n)$  respectively.

By (1.1.1) we have canonical morphisms

$$(1.1.2) \quad H_c^j(X, \mathbb{Q}_l) \rightarrow \text{IH}_c^j(X, \mathbb{Q}_l), \quad \text{IH}^{2n-j}(X, \mathbb{Q}_l)(n) \rightarrow H_j^{\text{BM}}(X, \mathbb{Q}_l),$$

which are dual of each other. Here  $H_j^{\text{BM}}(X, \mathbb{Q}_l) := H^{-j}(X, \mathbb{D}\mathbb{Q}_l)$  denotes (so-called) Borel-Moore homology which is the dual of  $H_c^j(X, \mathbb{Q}_l)$ .

**1.2. Weight filtration.** If  $k$  is an algebraic closure of a finite field, then there is the weight filtration  $W$  on  $H^j(X, \mathbb{Q}_l)$ ,  $H_c^j(X, \mathbb{Q}_l)$  and hence on their dual  $H_j(X, \mathbb{Q}_l)$ ,  $H_j^{\text{BM}}(X, \mathbb{Q}_l)$ . Moreover,  $H_c^j(X, \mathbb{Q}_l)$  and  $H_j^{\text{BM}}(X, \mathbb{Q}_l)$  have weights  $\leq j$  and  $\geq -j$  respectively, i.e.  $\text{Gr}_i^W H_c^j(X, \mathbb{Q}_l) = \text{Gr}_{-i}^W H_j^{\text{BM}}(X, \mathbb{Q}_l) = 0$  for  $i > j$ , see [2], [7].

It is known that these can be generalized to the case of any algebraically closed field. Indeed, if  $k$  has characteristic 0, then this is well known as part of the theory

of mixed motives via realizations, see [17]. The argument is similar for  $\text{char } k > 0$ . In this case, we may assume that  $X$  is defined over a subfield  $k'$  of  $k$  which is finitely generated over  $\overline{\mathbb{F}}_q$  where  $\overline{\mathbb{F}}_q$  is the algebraic closure of a finite field  $\mathbb{F}_q$  in  $k$ . Then there is a morphism  $\pi : \mathcal{X} \rightarrow S$  of algebraic varieties over  $\overline{\mathbb{F}}_q$  such that the function field of  $S$  is  $k'$  and the geometric generic fiber of  $\pi$  over the geometric generic point of  $S$  defined by the inclusion  $k' \rightarrow k$  is identified with  $X$ . Here we may assume that  $R^j\pi_*\mathbb{Q}_l$  and  $R^j\pi_!\mathbb{Q}_l$  are smooth sheaves shrinking  $S$  if necessary.

By [2] these smooth sheaves have the canonical weight filtration  $W$ , which is compatible with the weight filtration on  $H^j(\mathcal{X}_s, \mathbb{Q}_l)$  and  $H_c^j(\mathcal{X}_s, \mathbb{Q}_l)$  for any closed point  $s$  of  $S$  using the generic base change theorem [6]. Indeed, the stalk at  $s$  of the smooth pure perverse sheaf  $\mathcal{F}$  of weight  $r$  on  $S$  is pure of weight  $r - d$  with  $d = \dim S$ , because  $i_s^!\mathcal{F} = i_s^*\mathcal{F} \otimes i_s^!\mathbb{Q}_l$  for any smooth  $\mathbb{Q}_l$ -sheaf  $\mathcal{F}$  where  $i_s : \text{Spec } k \rightarrow S$  is defined by  $s$ . Restricting to the stalk at the geometric generic point, this induces the weight filtration  $W$  on  $H^j(X, \mathbb{Q}_l)$ ,  $H_c^j(X, \mathbb{Q}_l)$ , and then on  $H_j(X, \mathbb{Q}_l)$ ,  $H_j^{\text{BM}}(X, \mathbb{Q}_l)$  by duality. The obtained filtration is independent of the choice of  $k', S$ . This is functorial for morphisms of algebraic varieties in the usual way, because any morphism over  $k$  has a model over  $S$  replacing  $S$  if necessary.

**1.3. Remark.** Let  $\pi : \mathcal{X} \rightarrow S$  be as above. Then the pull-back of the intersection complex  $\text{IC}_{\mathcal{X}}\mathbb{Q}_l$  by the canonical morphism  $X \rightarrow \mathcal{X}$  is naturally isomorphic to  $\text{IC}_X\mathbb{Q}_l$  up to a shift of complexes. This can be shown by using the intermediate direct image [2] together with the generic base change theorem [6]. As a corollary we get the purity of intersection cohomology in the proper case using the above construction of weight filtration.

## 2. Proof of the main theorems

**2.1. Proof of Theorem 1.** Note that the canonical morphisms in Theorem 1 is induced by (1.1.2) together with the restriction morphism (and its dual). We first reduce the assertion to the case where  $k$  is an algebraic closure of a finite field  $k_0$ . In the case  $\text{char } k = p > 0$ , this follows from Remark (1.3) taking a closed point of  $S$ , because the canonical morphism

$$\text{Gr}_W^j H_c^j(X, \mathbb{Q}_l) \rightarrow \text{IH}^j(\overline{X}, \mathbb{Q}_l)$$

can be extended to a morphism of local systems

$$\text{Gr}_W^j R^j\pi_!\mathbb{Q}_l \rightarrow R^j\overline{\pi}_!(\text{IC}_{\overline{\mathcal{X}}}\mathbb{Q}_l),$$

where  $\overline{\pi} : \overline{\mathcal{X}} \rightarrow S$  is a compactification of  $\pi$  whose geometric generic fiber is  $\overline{X}$ . If  $\text{char } k = 0$ , the argument is similar using the mod  $p$  reduction argument in [2]. So we may assume that  $k$  is the algebraic closure of a finite field.

Let  $K$  denote the mapping cone  $C(j^!\mathbb{Q}_l \rightarrow \text{IC}_{\overline{X}}\mathbb{Q}_l)$ , where  $j' : X \rightarrow \overline{X}$  denotes the inclusion morphism. By the associated long exact sequence

$$\rightarrow H^{j-1}(\overline{X}, K) \rightarrow H_c^j(X, \mathbb{Q}_l) \rightarrow \text{IH}^j(\overline{X}, \mathbb{Q}_l) \rightarrow,$$

it is sufficient to show that  $H^{j-1}(\overline{X}, K)$  has weights  $\leq j - 1$ . By [2], 5.1.14, this is reduced to the assertion that  $K$  has weights  $\leq 0$  in the sense of [2], 5.1.8. By

definition this is equivalent to the condition that  $\mathcal{H}^j K_x$  has weights  $\leq j$  for any closed point  $x$  of  $\overline{X}$ . But this is further equivalent to the same assertion with  $K$  replaced by  $\mathrm{IC}_{\overline{X}}\mathbb{Q}_l$ , because  $\mathcal{H}^j K_x = \mathcal{H}^j(\mathrm{IC}_{\overline{X}}\mathbb{Q}_l)_x/\mathbb{Q}_l$  if  $x \in X$ ,  $j = 0$ , and  $\mathcal{H}^j K_x = \mathcal{H}^j(\mathrm{IC}_{\overline{X}}\mathbb{Q}_l)_x$  otherwise. (The isomorphism for  $j = 0$  follows from the fact that the constant sheaf  $\mathbb{Q}_l$  has no nontrivial subsheaf supported on a closed subvariety of strictly smaller dimension, which we apply to the kernel of  $\mathcal{H}^0$  of the first morphism in (1.1.1).) Then the above assertion for the intersection complex  $\mathrm{IC}_{\overline{X}}\mathbb{Q}_l$  is known as Gabber's purity theorem [2]. This completes the proof of Theorem 1.

**2.2. Remarks.** (i) The canonical morphism  $\mathrm{IH}^j(\overline{X}) \rightarrow \mathrm{Gr}_j^W \mathrm{IH}^j(X)$  is surjective. This follows from the distinguished triangle

$$i_* i^! \mathrm{IC}_{\overline{X}}\mathbb{Q}_l \rightarrow \mathrm{IC}_{\overline{X}}\mathbb{Q}_l \rightarrow \mathbf{R}j'_* \mathrm{IC}_X\mathbb{Q}_l \xrightarrow{+1},$$

(where  $i : \overline{X} \setminus X \rightarrow \overline{X}$  is the inclusion of the complement) together with the assertion that  $i^! \mathrm{IC}_{\overline{X}}\mathbb{Q}_l$  has weights  $\geq 0$ , see [2], 5.1.14.

(ii) If  $\overline{X}$  is projective and  $\mathrm{char} k = 0$ , then we have a canonical choice of a splitting of the surjective or injective morphisms in Theorem 1 using a polarization of Hodge structure. However, it is not clear whether it is a good one. For example, certain algebraic cycle classes (i.e. the graph of 'placid' maps) can be canonically lifted to the intersection cohomology in [11] (see also [1]), and the relation with this is unclear. This problem of canonical lifting is related to the Lefschetz trace formula for intersection cohomology (see [9], [11], [12], [13], [15], [16], [18], [19], [20], [23]), and will be treated in [8].

**2.3. Proof of Theorem 2.** Let  $K = C(\mathbb{Q}_l \rightarrow \mathrm{IC}_Y\mathbb{Q}_l)$ . Using the distinguished triangle

$$(2.3.1) \quad f^* K[-1] \rightarrow \mathbb{Q}_l \rightarrow f^* \mathrm{IC}_Y\mathbb{Q}_l \xrightarrow{+1},$$

the assertion is equivalent to the vanishing of the composition of canonical morphisms

$$(2.3.2) \quad f^* K[-1] \rightarrow \mathbb{Q}_l \rightarrow \mathrm{IC}_X\mathbb{Q}_l \quad \text{in } D_c^b(X, \mathbb{Q}_l).$$

Then the assertion is reduced to the case where  $k$  is an algebraic closure of a finite field. Indeed, in the positive characteristic case, we have

$$\mathrm{Hom}(f^* K[-1], \mathrm{IC}_X\mathbb{Q}_l) = H^1(X, \mathbf{R}\mathcal{H}om(f^* K, \mathrm{IC}_X\mathbb{Q}_l)),$$

and this vector space can be extended to a local system (i.e. a smooth sheaf) on  $S$  as in (1.2). Note that (2.3.2) is extended to a section of this local system and its vanishing is equivalent to that for its restriction over a closed point using the generic base change theorem [6]. (If  $\mathrm{char} k > 0$ , it is also possible to replace  $X$  with  $\mathcal{X}$  in (1.3).) In the case  $\mathrm{char} k = 0$ , this follows from the mod  $p$  reduction argument in [2].

By the same argument as in (2.1),  $f^* K$  has weights  $\leq 0$ . There is a finite subfield  $k_0$  of  $k$  together with a morphism  $f_0 : X_0 \rightarrow Y_0$  of algebraic varieties over  $k_0$  such that  $f$  is the base change of  $f_0$  and the above morphisms are defined over  $X_0$ .

Then the assertion follows from [2], 5.1.15 (applied to  $f^*K$  and  $\mathrm{IC}_X\mathbb{Q}_l$  which have weights  $\leq 0$  and  $\geq 0$  respectively with the normalization of intersection complexes in [1]). This completes the proof of Theorem 2.

The following was proved in [1] using induction on stratum at least if  $\mathrm{char} k = 0$ . We give another proof using the weight filtration  $W$  on mixed perverse sheaves [2].

**2.4. Proposition.** *Assume  $X, Y$  irreducible,  $f : X \rightarrow Y$  is a closed immersion, and  $\dim Y = \dim X + 1 (= n + 1)$ . Then there is a canonical choice of the morphism  $f^*\mathrm{IC}_Y\mathbb{Q}_l \rightarrow \mathrm{IC}_X\mathbb{Q}_l$  satisfying the condition in Theorem 2.*

*Proof.* Let  $j : Y \setminus X \rightarrow Y$  denote the inclusion morphism. We have the long exact sequence

$$\rightarrow {}^p\mathcal{H}^i\mathrm{IC}_Y\mathbb{Q}_l \rightarrow f_* {}^p\mathcal{H}^i f^*\mathrm{IC}_Y\mathbb{Q}_l \rightarrow {}^p\mathcal{H}^{i+1} j_* j^*\mathrm{IC}_Y\mathbb{Q}_l \rightarrow,$$

where  ${}^p\mathcal{H}^{i+1} j_* j^*\mathrm{IC}_Y\mathbb{Q}_l = 0$  for  $i + 1 > n + 1 (= \dim Y)$ , see [2], 4.2.4. So we get

$${}^p\mathcal{H}^i f^*\mathrm{IC}_Y\mathbb{Q}_l = 0 \quad \text{for } i > n,$$

because the perverse sheaf  $\mathrm{IC}_Y\mathbb{Q}_l[n + 1]$  has no nontrivial quotient whose support is contained in  $X$ . Furthermore,  $f^*\mathrm{IC}_Y\mathbb{Q}_l[n]$  and hence  ${}^p\mathcal{H}^n f^*\mathrm{IC}_Y\mathbb{Q}_l$  (see [2], 5.4.1) have weights  $\leq n$ . Thus we get a canonical morphism

$$f^*\mathrm{IC}_Y\mathbb{Q}_l[n] \rightarrow \mathrm{Gr}_n^W {}^p\mathcal{H}^n f^*\mathrm{IC}_Y\mathbb{Q}_l,$$

(factoring through  ${}^p\mathcal{H}^n f^*\mathrm{IC}_Y\mathbb{Q}_l$ ). By the semisimplicity theorem ([2], 5.3.8), the target of the above morphism is semisimple, and is a direct sum of a subobject whose support is strictly smaller than  $X$  and an intersection complex with support  $X$  associated with a smooth sheaf defined on a dense open subvariety of  $X$ . So a morphism

$$(2.4.1) \quad f^*\mathrm{IC}_Y\mathbb{Q}_l \rightarrow \mathrm{IC}_X\mathbb{Q}_l,$$

is uniquely determined by its restriction to any sufficiently small nonempty open subvariety  $X'$  of  $X$  (contained in the smooth part of  $X$ ), because  $\mathrm{IC}_X\mathbb{Q}_l[n]$  is a perverse sheaf and is pure of weight  $n$ . We have a similar assertion for a morphism  $\mathbb{Q}_l \rightarrow \mathrm{IC}_X\mathbb{Q}_l$ .

Let  $f' : X' \rightarrow Y'$  be the restriction (or base change) of  $f$  over an open subvariety  $Y'$  of  $Y$  whose complement has codimension  $\geq 2$ . Let  $\rho' : \tilde{Y}' \rightarrow Y'$  be the normalization, and  $X'' = (\tilde{Y}' \times_{Y'} X')_{\mathrm{red}}$  with the canonical morphism  $\rho'' : X'' \rightarrow X'$ . Let  $X''_i$  be the irreducible components of  $X''$  with  $s_i$  the separable degree of the extension  $k(X''_i)/k(X')$ . Here we may assume that  $\tilde{Y}'$  and  $X''$  are smooth over  $k$ , and  $f'^*\rho'_*\mathbb{Q}_l = \rho''_*\mathbb{Q}_l$  is a local system (shrinking  $Y'$  if necessary). Then we have

$$\mathrm{IC}_{Y'}\mathbb{Q}_l = \rho'_*\mathbb{Q}_l, \quad f^*\mathrm{IC}_Y\mathbb{Q}_l = \rho''_*\mathbb{Q}_l,$$

and the restriction of the morphism (2.4.1) to  $X'$  is given by  $\rho''_*\mathbb{Q}_l \rightarrow \mathbb{Q}_l$ .

Let  $X_i^s$  be the irreducible variety with finite morphisms  $X_i'' \rightarrow X_i^s \rightarrow X'$  factoring  $\rho''$  so that the function field  $k(X_i^s)$  is the maximal separable extension of  $k(X')$  contained in  $k(X_i'')$  (shrinking  $X'$  if necessary). Then the direct image of the constant sheaf  $\mathbb{Q}_l$  by  $X_i'' \rightarrow X_i^s$  is the constant sheaf  $\mathbb{Q}_l$  so that we may replace  $\rho''_*\mathbb{Q}_l$  with  $\rho_*^s\mathbb{Q}_l$  where  $\rho^s$  is the canonical morphism of the disjoint union of the  $X_i^s$

to  $X'$ . So the desired morphism  $\rho_*^s \mathbb{Q}_l \rightarrow \mathbb{Q}_l$  is given by the dual of the canonical morphism  $\iota : \mathbb{Q}_l \rightarrow \rho_*^s \mathbb{Q}_l$ , divided by  $\sum_i s_i$ , because the composition of  $\iota$  and its dual  $\iota^\vee : \rho_*^s \mathbb{Q}_l \rightarrow \mathbb{Q}_l$  is the multiplication by  $\sum_i s_i$  on  $\mathbb{Q}_l$ . This completes the proof of Proposition (2.4).

**2.5. Remarks.** (i) Even in the case of Proposition (2.4), the morphism satisfying the condition in Theorem 2 is not unique in general (for example, if  $Y$  has etale locally two components whose intersection is  $X$ ). However, it is unique if  $Y$  is etale locally irreducible at the generic point of  $X$ , see also [1].

(ii) In the case  $f$  is a closed immersion of codimension  $\geq 2$ , the morphism satisfying the condition in Theorem 2 is not unique even if  $X, Y$  have only isolated singularities. For simplicity, assume  $\dim Y = \dim X + 2 (= n + 2)$ . For  $x \in \text{Sing } X$ , let

$$E_{X,x}^i = \mathcal{H}^i(\text{IC}_X \mathbb{Q}_l)_x, \quad E_{Y,x}^i = \mathcal{H}^i(\text{IC}_Y \mathbb{Q}_l)_x.$$

Then we have a distinguished triangle

$${}^p\tau_{\leq n} f^* \text{IC}_Y \mathbb{Q}_l \rightarrow f^* \text{IC}_Y \mathbb{Q}_l \rightarrow E_{Y,x}^{n+1}[-n-1] \xrightarrow{+1},$$

and there is a contribution of

$$\text{Hom}(E_{Y,x}^{n+1}[-n-1], \text{IC}_X \mathbb{Q}_l),$$

to the ambiguity of the morphism using the above distinguished triangle together with the vanishing of negative extensions. Moreover, by the self duality

$$\mathbb{D} \text{IC}_X \mathbb{Q}_l = \text{IC}_X \mathbb{Q}_l(n)[2n],$$

the last group is isomorphic to

$$\begin{aligned} & \text{Hom}(\text{IC}_X \mathbb{Q}_l, (\mathbb{D} E_{Y,x}^{n+1})(-n)[-n+1]) \\ &= \text{Hom}(E_{X,x}^{n-1}, (\mathbb{D} E_{Y,x}^{n+1})(-n)), \end{aligned}$$

which is not necessarily zero in general. For example, if  $X$  is the affine cone of a smooth projective variety  $V$ , then  $E_{X,x}^j$  is isomorphic to the primitive cohomology  $H_{\text{prim}}^j(V, \mathbb{Q}_l)$ .

### 3. Application of the decomposition theorem

**3.1. Decomposition theorem.** Let  $f : X \rightarrow Y$  be a proper surjective morphism of irreducible algebraic varieties over an algebraically closed field  $k$ . Let  $n = \dim X$ . By the decomposition theorem [2], we have a noncanonical isomorphism

$$(3.1.1) \quad \mathbf{R}f_* \text{IC}_X \mathbb{Q}_l[n] \simeq \bigoplus_{i,Z} \text{IC}_Z E_{Z^\circ}^i[\dim Z][i] \quad \text{in} \quad D_c^b(Y, \mathbb{Q}_l),$$

where  $Z$  are irreducible reduced closed subvarieties of  $Y$ , and  $E_{Z^\circ}^i$  are smooth  $l$ -adic sheaves defined on dense open smooth subvarieties  $Z^\circ$  of  $Z$  which we may assume to be independent of  $i$ . This can be reduced to the case where the base field  $k$  is finite by using a model as in (1.2) together with [6]. Note that the shift by  $n$  or

$\dim Z$  is needed for  $\mathrm{IC}_X \mathbb{Q}_l$ ,  $\mathrm{IC}_Z E_{Z^\circ}^i$  because of the normalization of the intersection complex in this paper. Set

$$m_Z = \max\{i \mid E_{Z^\circ}^i \neq 0\} = \max\{i \mid E_{Z^\circ}^{-i} \neq 0\}.$$

Note that  $E_{Z^\circ}^i$  is the dual of  $E_{Z^\circ}^{-i}$  up to a Tate twist.

**3.2. Lemma.** *With the above notation, assume  $X$  is smooth over  $k$  so that  $\mathrm{IC}_X \mathbb{Q}_l = \mathbb{Q}_l$ . Then*

$$(3.2.1) \quad m_Z \leq n - \dim Z - 2 \text{ if } Z \neq Y, \quad m_Y \leq n - \dim Y.$$

*Proof.* Let  $X_s = f^{-1}(s)$ . For a general closed point  $s$  of  $Z^\circ$ , we have

$$m_Z \leq 2 \dim X_s + \dim Z - n,$$

because the stalk of  $R^j f_*(\mathbb{Q}_l[n])$  at  $s$  vanishes unless  $j \in [-n, 2 \dim X_s - n]$  and the stalk of  $\mathrm{IC}_Z E_{Z^\circ}^i[\dim Z] [-i]$  at  $s$  is  $E_{Z^\circ, s}^i$  put at the degree  $j = i - \dim Z$ . We have  $\dim X_s \leq n - \dim Z - 1$  if  $Z \neq Y$ . So the assertion follows.

As a corollary we have the following (which is used in [8]).

**3.3. Proposition.** *With the notation of (3.1), assume  $X$  smooth. Let*

$$u : \mathbb{Q}_l[n] \rightarrow \mathbf{R}f_* \mathbb{Q}_l[n], \quad v : \mathbf{R}f_* \mathbb{Q}_l[n] \rightarrow \mathbb{D}\mathbb{Q}_l(-n)[-n]$$

*be the canonical morphisms on  $Y$ , where  $v$  is the dual of  $u$ . Let*

$$(3.3.1) \quad \begin{aligned} u_{Z,i} : \mathbb{Q}_l[n] &\rightarrow \mathrm{IC}_Z E_{Z^\circ}^i[\dim Z] [-i], \\ v_{Z,i} : \mathrm{IC}_Z E_{Z^\circ}^i[\dim Z] [-i] &\rightarrow \mathbb{D}\mathbb{Q}_l(-n)[-n] \end{aligned}$$

*be the induced morphisms using the decomposition (3.1.1). Then*

$$(3.3.2) \quad \begin{aligned} u_{Z,i} &= 0 \text{ for } (Z, i) \neq (Y, n - \dim Y), \\ v_{Z,i} &= 0 \text{ for } (Z, i) \neq (Y, \dim Y - n). \end{aligned}$$

*Proof.* By the adjunction for the inclusion  $Z \rightarrow Y$ , we have the canonical isomorphism

$$\begin{aligned} &\mathrm{Hom}_Y(\mathbb{Q}_l[n], \mathrm{IC}_Z E_{Z^\circ}^i[\dim Z] [-i]) \\ &= \mathrm{Hom}_Z(\mathbb{Q}_l[\dim Z], \mathrm{IC}_Z E_{Z^\circ}^i[\dim Z] [\dim Z - n - i]), \end{aligned}$$

and the last group vanishes for  $\dim Z - n - i < 0$  by the semi-perversity of the constant sheaf  $\mathbb{Q}_l[\dim Z]$  on  $Z$ . So we get the first assertion by (3.2.1). The assertion is similar for the second.

As a corollary of (3.3), we have

**3.4. Corollary.** *Let  $f : X \rightarrow Y$  be a proper surjective morphism of irreducible algebraic varieties over an algebraically closed field  $k$ . Assume  $X$  is smooth over*



*k*. Then

$$\begin{aligned} \operatorname{Ker}(H^i(Y, \mathbb{Q}_l) \rightarrow H^i(X, \mathbb{Q}_l)) &= \operatorname{Ker}(H^i(Y, \mathbb{Q}_l) \rightarrow \operatorname{IH}^i(Y, \mathbb{Q}_l)), \\ \operatorname{Ker}(H_c^i(Y, \mathbb{Q}_l) \rightarrow H_c^i(X, \mathbb{Q}_l)) &= \operatorname{Ker}(H_c^i(Y, \mathbb{Q}_l) \rightarrow \operatorname{IH}_c^i(Y, \mathbb{Q}_l)), \\ \operatorname{Im}(H_i(X, \mathbb{Q}_l) \rightarrow H_i(Y, \mathbb{Q}_l)) &= \operatorname{Im}(\operatorname{IH}_i(Y, \mathbb{Q}_l) \rightarrow H_i(Y, \mathbb{Q}_l)), \\ \operatorname{Im}(H_i^{\operatorname{BM}}(X, \mathbb{Q}_l) \rightarrow H_i^{\operatorname{BM}}(Y, \mathbb{Q}_l)) &= \operatorname{Im}(\operatorname{IH}_i^{\operatorname{BM}}(Y, \mathbb{Q}_l) \rightarrow H_i^{\operatorname{BM}}(Y, \mathbb{Q}_l)). \end{aligned}$$

*Proof.* This follows from (3.3).

**3.5. Remarks.** (i) The above corollary is related to a question of A. Weber when  $X \rightarrow Y$  is a resolution of singularities.

(ii) The first isomorphism of (3.4) implies a proof of Theorem 1 in the case  $k = \mathbb{C}$  and  $X$  proper, using the weight spectral sequence in [5]. Indeed, if  $X \rightarrow Y$  is a resolution of singularities, then  $\operatorname{Gr}_i^W H^i(Y)$  is a subspace of  $H^i(X)$  (which coincides with the image of  $H^i(Y)$ ) by the weight spectral sequence associated to a simplicial resolution as in Remark (iii) below. So we can replace  $H^i(X)$  with  $\operatorname{IH}^i(Y)$  by (3.4).

(iii) Let  $Y$  be a proper irreducible variety of dimension  $n$  over an algebraically closed field  $k$ . Consider the natural morphisms

$$\operatorname{Gr}_n^W H^n(Y, \mathbb{Q}_l) \xrightarrow{u_n} \operatorname{IH}^n(Y, \mathbb{Q}_l) \xrightarrow{v_n} \operatorname{Gr}_n^W (H_n(Y, \mathbb{Q}_l)(-n)),$$

where  $u_n$  is injective and  $v_n$  is the dual of  $u_n$  and is surjective. These are induced by the morphisms  $u, v$  in (3.3). We have a canonical self-pairing of  $\operatorname{IH}^n(Y, \mathbb{Q}_l)$ , and a canonical pairing between  $H^n(Y, \mathbb{Q}_l)$  and  $H_n(Y, \mathbb{Q}_l)(-n)$  with values in  $\mathbb{Q}_l(-n)$  so that

$$(3.5.1) \quad \langle u(\xi), \eta \rangle = \langle \xi, v(\eta) \rangle \quad \text{for } \xi \in H^n(Y, \mathbb{Q}_l), v \in \operatorname{IH}^n(Y, \mathbb{Q}_l).$$

By (3.3) and (3.4) we can replace  $\operatorname{IH}^n(Y, \mathbb{Q}_l)$  with  $H^n(X, \mathbb{Q}_l)$  where  $X$  is a resolution of singularities of  $Y$ . By (3.1.1)  $\operatorname{IH}^n(Y, \mathbb{Q}_l)$  is a direct factor of  $H^n(X, \mathbb{Q}_l)$  noncanonically, and the morphisms  $u$  and  $v$  are compatible with any decomposition (3.1.1) by (3.3).

(iv) With the above notation, the restriction of the self-pairing to the image of  $u_n$  does not seem to be nondegenerate in general. Indeed, if we take a simplicial resolution  $X_\bullet$  of  $Y$  such that  $X_0 = X$  in the case of  $k = \mathbb{C}$ , then this image coincides with the kernel of  $H^n(X) \rightarrow H^n(X_1)$ . However, this does not seem to be compatible with the Lefschetz decomposition in general.

For example, let  $X$  be the blow-up of  $\mathbb{P}^2$  along a point. This is a  $\mathbb{P}^1$ -bundle over  $C = \mathbb{P}^1$  having disjoint two sections  $C_i$  whose self-intersection number is  $i$  for  $i = \pm 1$ . If there is a variety  $Y$  which is obtained by identifying the two sections  $C_1$  and  $C_{-1}$ , then the restriction of the canonical pairing to the kernel of  $H^2(X) \rightarrow H^2(C)$  would be degenerate, where the last morphism is defined by the difference between the restriction morphisms to  $C_1$  and  $C_{-1}$  (both identified with  $C$  naturally). However, it is not clear if such  $Y$  exists in the category of algebraic varieties (although it exists as an analytic space if  $k = \mathbb{C}$ ). It would not be projective at least.

**3.6. Remark.** Let  $f : X \rightarrow Y$  be a proper surjective morphism of irreducible varieties over an algebraically closed field  $k$  as in (3.1). Let  $n = \dim X$ . Assume

$$(3.6.1) \quad \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l[n] \text{ is a perverse sheaf on } Y.$$

This assumption implies that  $\dim X = \dim Y = n$ , and the direct sum decomposition (3.1.1) becomes the canonical decomposition in the category of perverse sheaves

$$(3.6.2) \quad \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l[n] = \bigoplus_Z \mathrm{IC}_Z E_{Z^\circ}[\dim Z],$$

because there are no nontrivial morphisms between intersection complexes with different supports. This induces canonical morphisms

$$\mathrm{IC}_Y \mathbb{Q}_l \rightarrow \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l, \quad \mathbf{R}f_* \mathrm{IC}_X \mathbb{Q}_l \rightarrow \mathrm{IC}_Y \mathbb{Q}_l.$$

Indeed, over a sufficiently small non-empty open subvariety of  $Y$ , the intersection complexes coincide with the constant sheaf  $\mathbb{Q}_l$ , and the assertion is clear. Then we can extend the obtained morphisms uniquely over  $Y$  using the decomposition (3.6.2) together with the intermediate direct image [2]. Here we can neglect  $\mathrm{IC}_Z E_{Z^\circ}$  for  $Z \neq Y$ , because an intersection complex has no nontrivial sub nor quotient objects with strictly smaller support.

The condition (3.6.1) is satisfied for example if  $\dim X = \dim Y = 2$ , or  $f$  is finite and surjective. In the first case, (3.6.1) follows from the fact that the support of  $\mathcal{H}^1 \mathrm{IC}_X \mathbb{Q}_l$  is discrete. In the second case, the direct image is an intersection complex, because the direct image by a finite morphism is an exact functor of perverse sheaves, see [2].

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